Complex Analysis: Midterm Exam

Aletta Jacobshal 01, Monday 17 December 2018, 09:00–11:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of each of your exam sheets and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you **must** explain why the conditions for using such results are satisfied.
- 10 points are "free". There are 5 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.

Question 1 (15 points)

Prove that if a function $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is differentiable at $z_0 = x_0 + \mathrm{i}y_0$ then the derivative is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + \mathrm{i}\frac{\partial v}{\partial x}(x_0, y_0).$$

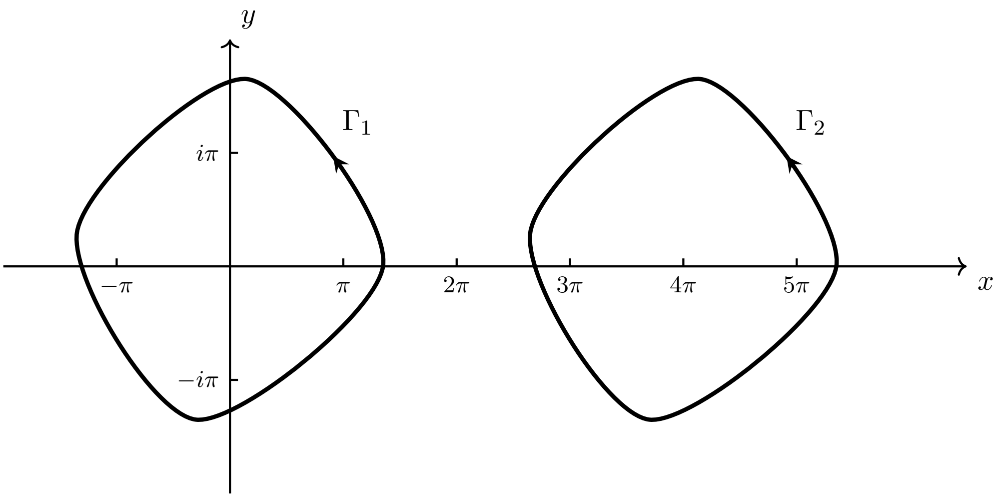
Question 2 (20 points)

Prove that the function $f(z) = \sqrt{z+1}\sqrt{z-1}$ is discontinuous at $z \in (-1, 1)$, that is, along the open interval on the real axis between -1 and 1. You **must** check the limits of $f(z)$ at $z \in (-1, 1)$.

Here $\sqrt{z}$ is the principal value $e^{\frac{1}{2}\operatorname{Log} z}$ of the multi-valued $z^{1/2}$. It is given that for $-r < 0$ we have $\lim_{t\to 0^+} \sqrt{-r + \mathrm{i}t} = \mathrm{i}\sqrt{r}$ and $\lim_{t\to 0^-} \sqrt{-r + \mathrm{i}t} = -\mathrm{i}\sqrt{r}$.

Question 3 (20 points)

Compute $\displaystyle\int_\Gamma \frac{ze^z}{(z - \mathrm{i}\pi)^2}\,\mathrm{d}z$ first for the contour $\Gamma = \Gamma_1$ and then for the contour $\Gamma = \Gamma_2$ shown below.



Question 4 (20 points)

Prove that on the positively oriented circle C given by $|z+1| = 2$ we have

$$\left| \int_C \frac{e^z}{\bar{z} - 3} \, \mathrm{d}z \right| \leq 2\pi e.$$

NB: The inequalities that you use will either need to be proved (algebraically or geometrically) or to be part of the theory presented in the book. In particular, the triangle inequality, in its different forms, is considered known and so are inequalities between $\operatorname{Re} z$, $\operatorname{Im} z$ and $|z|$. Fewer points will be given if the necessary inequalities are established only "visually" (that is, by looking at the right picture but without complete proof).

Question 5 (15 points)

A function $f(z)$ on $\mathbb{C}$ is **doubly periodic** if there are non-zero complex numbers ω_1 and ω_2 such that:

(a) $f(z + \omega_1) = f(z + \omega_2) = f(z)$ for all $z \in \mathbb{C}$, and

(b) ω_1 and ω_2 are linearly independent over the reals, that is, $\omega_2/\omega_1 \notin \mathbb{R}$, implying that each $z \in \mathbb{C}$ can be written as $z = \lambda_1\omega_1 + \lambda_2\omega_2$ with unique $\lambda_1, \lambda_2 \in \mathbb{R}$.

Prove that if a doubly periodic function is entire then it must be constant.

Hint: Suppose that $g : V \to \mathbb{R}$, where V is a closed and bounded subset of $\mathbb{C}$, is continuous. It is then known that there is $M > 0$ such that $-M \leq g(z) \leq M$ for all $z \in V$.

Formulas

The Cauchy-Riemann equations for a function $f = u + \mathrm{i}v$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The principal value of the logarithm is

$$\operatorname{Log} z = \operatorname{Log} |z| + \mathrm{i} \operatorname{Arg} z.$$

The generalized Cauchy integral formula is

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$